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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 4 — BATON ROUGE, LA., JUNE, 1930 — NO. 8

ROUNDING OUT OUR FOURTH YEAR

With this issue of the Mathematics News Letter is completed the fourth year of its existence. Originally but a small four-page leaflet mailed out from Baton Rouge to mathematics teachers in Louisiana and Mississippi free of cost to them (the expense of printing and distributing it was largely borne by voluntary contributions), in its first year it functioned as a mere campaign folder. Its only object in that year (the year 1926-27) was to arouse and develop the interest of our Mississippi and Louisiana mathematical workers in a proposed joint meeting of secondary and college representatives in the two States for the purpose of establishing a permanent cooperative relation between the National Council of Teachers of Mathematics and the Mathematical Association of America within Louisiana-Mississippi territory.

This meeting, held in Shreveport in March, is now a matter of history. With nationally known figures present and contrib-

uting to the program, as well as aiding in the task of bringing about the proposed consolidation, there was effected as a result of the meeting the scheme of cooperation which is now being carried out by high school and college mathematical workers of the two States.

Then the idea was conceived of expanding the little four-page campaign leaflet into a permanent Mathematics News Letter. It would help to keep alive the professional enthusiasms and ideals easily and naturally generated by the contacts of the annual joint meetings of the Council Branch and M. A. of A. Section. It would be a medium of communication between the national organizations and the local branches of mathematical activity. It should fan the interest of school and college administrations in the cultural and disciplinary aspects of mathematics. It could be a clearing-house of professional information for all mathematical people residing in the home territory. The timid beginner and interested student could alike use its pages for their mathematical self-expression. Of equal value to all this, teachers of grade, secondary and college mathematics with their mutual and correlated problems, vexing and numerous as these have always been, could use the pages of the Letter as a means of unifying their diversities of view-point and discovering a solid common teaching ground.

Has the idea been justified in the sequel? Much space might be used in writing to the question. To do so would profit little. Few besides the writer know the true measure of the handicaps and obstructions that have beset the path of the little journal. Its promoters have made utmost effort to have its pages reflect, at least in all essential respects, the aims outlined above and more broadly expressed on the title page. But we are keenly conscious that in many respects it is in need of improvement. We mention but one. We refer to the need of a greater balance between the two types of contributed material, namely, the high school and the college material. Much of the Letter's force as a joint organ of these two classes of mathematical workers is lost when for any cause the authorship of its content falls exclusively within one or the other of the two classes

rather than in both of them. Though this need is not our greatest one, the future is not void of a rosy hue. Our **subscribers are now found in 28 states** of the Union.

—*S.T.S.*

MATHEMATICS IN THE HIGH SCHOOL LIBRARY

We need to stimulate the interest of the high school student of mathematics beyond the point of a mere passing grade. We should strive to create an interest in mathematics that will cause him to pursue the study throughout his college course and even beyond the college.

This can be accomplished by giving him contact with facts from the history of mathematics and facts concerning the contribution mathematics has made to civilization. If a student never becomes acquainted with the real value of mathematics he derives from actual study of the Algebra and Geometry text books it is quite likely that he will not see the greatness of the subject.

Mr. Glenn Frank, in speaking of the men of high calibre leaving the laboratories of the American Universities for the laboratories of American industry, said that something must be done to prevent the well-springs of productive scholarship in the sciences from drying up at their source.

Even those that are loudest in their claim that too much mathematics is required in the high schools would not dare to argue that the high school should fail to prove a source of supply of students of higher mathematics.

The editors of the News Letter are making a special effort to render a service to the high schools of Louisiana and Mississippi by putting the News Letter in every high school library. The News Letter could be used to advantage in the high school library as reference material, and every student in high school could have opportunity to see what mathematics has contributed to civilization and to just what extent the civilization of today rests upon mathematics.

—*H. S.*

COMMENT INVITED

The News Letter is in receipt of the list of questions given below, which questions were assigned to a class of freshmen in a certain institution of higher learning in Louisiana. The students concerned were high school graduates and, in addition, spent ten lessons reviewing the material covered by the test. Fifteen students took the test. The number of students giving correct answers is indicated in italics after each respective question.

Consider the questions, the number of correct answers, the fact that the students were high school graduates, and that they had had a good review immediately before taking the test; what is the reaction in the mind of those who teach mathematics? What comment is offered?

Test.

(Fifteen students took the test.)

I. Solve by completing the square $3x^2-7x+2=0$.
Seven correct answers.

II. Solve $\sqrt{x+12}=2+\sqrt{x}$.
Three correct answers.

III. Solve graphically $x^2+y^2=25$, $x+y=7$.
Four correct answers.

IV. Solve by simultating $x^2+y^2=17$, $xy=4$.
Five correct answers.

V. Rationalize the denominator of $\dfrac{1}{1+2\sqrt{-2}}$
One correct answer.

VI. Simplify:

(a) $\sqrt{-9}+2\sqrt{-4}-3\sqrt[3]{-27}$ *One correct answer.*

(b) $(\sqrt{-8})(\sqrt[3]{-8})(\sqrt{-1})$ *One correct answer.*

(c) $\dfrac{a+x^0y^{-2}}{b-y^0x^{-2}}$ *One correct answer.*

(d) i^{101}

VII. Write the 12th term of $(1-\sqrt{x})^{21}$. *No correct answer.*

A PROBLEM IN INSURANCE

By I. C. NICHOLS
Louisiana State University

A, B, and C insure their respective houses in the same insurance company and at the same rate per thousand dollars per annum risk. A takes out a one-year policy and pays each year, in advance, $100.00 *premium; B takes out a three-year policy and pays, in advance,* $250.00 *premium; while C takes out a five-year policy, and pays, in advance,* $400.00 *premium; that is, B and C respectively take advantage of the customary discount of a half year for a three-year policy, and of a whole year for a five-year policy. Which form of policy is the cheaper? And how much cheaper?*

Discussion and solution: The business world is very familiar with the mathematical formula $S=P\ (1+i)^n$, wherein S represents the sum of money to which P dollars will accumulate in n years at the interest rate i.

Particular attention is called to the quantities n and i of this formula—the *time* and the *rate.* Even the most inexperienced business man knows that *time* and *rate* each have an immediate effect upon a loan of P dollars. Therefore when the insurance company gave B a discount of $50.00 and C a discount of $100.00, A did not accept this as prima facie evidence that they were given a better deal than he was given—A knew that *time* played a part in the transactions; and he knew, too, that the *rate* at which money could be bought must be considered. Hence the natural question, "Who is the best business man, A, B, or C? And why?"

Since policies are written only for one, three, or five years, it will be convenient to consider insurance written over a period of fifteen years, thus giving an exact multiple of one, three, and five, and hence enabling only full time policies of each form to be treated. Thinking then in terms of fifteen years, the cash equivalent or present value of each of the three forms of policies may be calculated and comparisons made of these cash values. Using the formula stated above $P=S/(I+i)^n$, and assuming money to be worth 6 per cent per annum interest, the

cash value of A's premiums, which are payable, in advance one year at a time, is

$$100+(100/1.06)+(100/1.06^2)$$
$$+(100/1.06^3)+\ldots\ldots+(100/1.06^{14})$$

The cash value of B's premiums, payable, in advance, at intervals of three years each, is

$$250+(250/1.06^3)+(250/1.06^9)+(250/1.06^{12}).$$

Similarly the cash value of C's premiums payable, in advance, at intervals of five years each, is

$$400+(400/1.06^5)+(400/1.06^{10})$$

Reducing these expressions, the cash values of the premiums paid by A, B, C over a period of fifteen years, interest on money being figured at 6 per cent per annum, are respectively \$1029.498, \$908.462, \$922.661.

Obviously the three-year policy is the cheaper if money be figured at 6%.

As the interest rate on money is increased the differences between A, B, C decrease both in magnitude and in percentage of magnitude. The corresponding results for 6%, 7%, 8%, 9% and 10% interest rates are shown respectively in the table below. The first column indicates the interest rates on money; columns A, B, and C indicate corresponding cash values for fifteen years of policies issued for one, three and five years respectively; and columns five and six indicate the percentage that A loses by taking out a one-year policy instead of a three or a five year policy respectively.

Interest rate	A	B	C	$\frac{A-B}{A}$	$\frac{A-C}{A}$
6............	1029.50	908.46	922.66	11.7	10.3
7............	974.55	867.65	888.43	11.0	8.9
8............	924.42	830.39	857.51	10.2	7.2
9............	878.62	796.10	828.94	9.4	5.5
10............	836.67	764.63	802.59	8.6	4.0

A survey of the table reveals that for ordinary rates of interest a three-year policy is more economical than a one-year or a five-year; that a five year policy is preferable to a one-year; and that, as interest rates increase, the preference of one policy over another tends to diminish.

ON BOUNDARY VALUES FOR AREA OF A TRIANGLE

By C. D. SMITH
Louisiana College

A familiar theorem of geometry states that two triangles with one angle in common are proportional to the sides of the angle. This theorem suggests the interesting question of possible boundary values for area when an angle is known. Such values would be useful in the study of sets of triangles when an angle is constant or when the perimeter is constant. It should also be interesting to know certain boundary values for area when both the perimeter and the angle vary. The purpose of the present paper is to develop a theorem relative to such boundary values.

Assume a triangle ABC ($A \leq B \leq C$) with area $\triangle$ and perimeter $2s$. We know that $\triangle = rs$ where r is the radius of the inscribed circle and also that $C = 180° - (A+B)$. Moreover from trigonometry we have well known equalities.

$$
\begin{aligned}
&\text{ctn } \tfrac{1}{2} A = (1+\cos A)/\sin A = (s-a)/r, \\
(1)\quad &\text{ctn } \tfrac{1}{2} B = (1+\cos B)/\sin B = (s-b)/r, \\
&\text{ctn } \tfrac{1}{2} C = (1+\cos C)/\sin C = (s-c)/r \\
&\qquad = \sin (A+B/[1+\cos (A+B)].
\end{aligned}
$$

The product of equation (1) gives

$$\frac{(1+\cos A)\ (1+\cos B)\ \sin (A+B)}{\sin A\ \sin B\ [1+\cos (A+B)\ ]} = \frac{(s-a)\ (s-b)\ (s-c)}{r^3} = \frac{s^2(s-a)\ (s-b)\ (s-c)}{r^3 s^2} = \frac{s^2}{\triangle}$$

In the special case when $A=B$ we have after reduction

$$(2)\qquad \frac{s^2}{\triangle} = \frac{(1+\cos A)^2}{\sin A\ \cos A} \text{ or } \frac{(1+\cos B)^2}{\sin B\ \cos B}$$

Now in the case when $A<B$ we next prove the interesting inequality

$$(3)\qquad \frac{(1+\cos A)^2}{\sin A\ \cos A} > \frac{s^2}{\triangle} > \frac{(1+\cos B)^2}{\sin B\ \cos B}$$

The proof of (3) is quite simple when expressed in terms of the sides by means of the laws of sines and cosines. However, the method here given though not so short seems to be more natural since the expressions used are functions of the angles A and B.

To prove the first inequality let

$$\frac{(1+\cos A)\ (1+\cos B)\ \sin\ (A+B)}{\sin A \sin B\ [1+\cos\ (A+B)\ ]} < \frac{(1+\cos\ A)^2}{\sin A\ \cos A}$$

or

$$\frac{(1+\cos\ B)\ \sin\ (A+B)}{\sin B\ \ [1+\cos\ (A+B)\ ]} < \frac{1+\cos\ A}{\cos\ A}$$

Next multiply by Cos A Sin B [1+Cos (A+B)], expand the functions of (A+B, transpose terms from right to left and simplify. We may then use the identities $\text{Cos}^2A=1-\text{Sin}^2A$ and $\text{Cos}^2B+\text{Sin}^2B=1$ and get the inequality

$\text{SinA CosA Cos B}-\text{Sin}^2\text{A SinB}+\text{SinA CosA}-$
$\text{SinB CosA CosB}+\text{SinA Sin}^2\text{B}-\text{CosA SinB}<0,$
or $\text{SinA}[\text{Cos}(A+B)+\text{CosA}]-\text{SinB}[\text{Cos}(A+B)+\text{CosA}]<0,$
or $\text{Cos}(A+B)+\text{CosA}>0$, since SinA—SinB is negative,
or $-\text{CosC}+\text{CosA}>0$, and $A<C$ as required.

From (2) and (3 we may now write the inequality

$$(4)\qquad s^2 \leqq \frac{\triangle}{\text{sinA}} \cdot \frac{(1+\text{cosA})^2}{\text{cosA}}$$

Derivative of $(1+\text{CosA})^2/\text{CosA}$ in respect to A is $\text{SinA Tan}^2\text{A}$ and hence the function is an increasing function of A from 0 to $\pi/3$, attaining the value 9/2 for $A=\pi/3$. By substituting 9/2 for $(1+\text{CosA})^2/\text{CosA}$ in (4) we set an upper bound for s^2. The result expressed as a lower bound for $\triangle$ is

$$(5)\qquad \triangle \geqq 2/9\ s^2 \sin A.$$

Referring again to (3) we may set

$$\frac{(1+\text{cosA})\ \ (1+\text{cosB})\ \ \sin(A+B)}{\text{sinA}\ \ \text{sinB}[1+\cos(A+B)]} > \frac{(1+\text{cosB})^2}{\text{sinB cosB}}$$

or
$$\frac{(1+\cos A)\ \sin(A+B)}{\sin A[1+\cos(A+B)]} > \frac{1+\cos B}{\cos B}$$

Now multiply by $\sin A \cos B[1+\cos(A+B)]$, expand functions of $(A+B)$, transpose terms from right to left and simplify. Then using the identities $\cos^2 B=1-\sin^2 B$ and $\cos^2 A+\sin^2 A=1$ we get

$\sin B \cos A \cos B - \sin A \sin^2 B + \sin B \cos B - \sin A \cos A \cos B + \sin^2 A \sin B - \sin A \cos B > 0$,

or $\sin B[\cos(A+B)+\cos B]-\sin A[\cos(A+B)+\cos B]>0$,

or $\cos(A+B)+\cos B>0$, since $\sin B - \sin A$ is positive,

or $-\cos C+\cos B>0$, and $B<C$ as required. From (2) and (3) we now have the inequality

$s^2 \geqq \triangle(1+\cos B)^2/\sin B \cos B$, or $\triangle \leqq s^2 \cos B \sin B/(1+\cos B)^2$, or (6) $\triangle \leqq s^2 \cos B$, since $\sin B/(1+\cos B)^2$ can not be greater than 1. Results (5) and (6) establish the Theorem:

If $\triangle$ is the area of the triangle ABC ($A \leqq B \leqq C$) of perimeter $2s$ the area is bounded by the inequality

$$2/9\ s^2 \sin A \leqq \triangle \leqq s^2 \cos B.$$

If we exclude the trivial case where $A=0$ the lower bound $2/9\ s^2 \sin A$ will be attained only for the equilateral triangle $A=\pi/3$. In case of the upper bound $s^2 \cos B$ the equality sign represents the limiting case where B approaches $\pi/2$.

Professors H. L. Smith and S. T. Sanders have discussed a bound for $\frac{s^2}{\triangle}$. Mathematics News Letter, Vol. 3, No. 8, April, 1929.

AN APPLICATION OF CERTAIN PARTIAL DERIVED FUNCTIONS

By F. A. RICKEY
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The Ampere-Cauchy Derived Functions.

Any function $f(x)$ of a single variable x gives rise to an associated function $f_1(x_0x_1)$ of two variables defined for all distinct values of x_0 and x_1 by the equation

$$f(x_0x_1)=[f(x_0)-f(x_1)]/(x_0-x_1)$$

Thus assuming that f_1, f_2, ----f_{n-1} have been inductively defined as functions of 2, 3, ---, n distinct arguments, respectively, we can then define f_n by the equation

$$f_n(x_0\text{---}x_n) = \frac{f_{n-1}(x_0\text{---}x_{n-1}) - f_{n-1}(x_1\text{---}x_n)}{x_0 - x_n}$$

for all distinct values of x_0, ---, x_n. The functions f_1, f_2, ---, f_n are known as the Ampere Cauchy derived functions of order 1, 2, ---, n, respectively. [See H. L. Smith's "On the Ampere-Cauchy Derived Functions," Annals of Mathematics, December, 1923.] In a paper not yet submitted for publication, Dr. Smith develops incidentally a formula for the nth derived function of a function g(t) defined by $g(t)=f(x)$, $x=h(t)$. The formula follows:

$$g_n(t_0\text{---}t_n) = \Sigma f_m(X_0\text{---}X_m)h(T_0\text{---}T_1)h(T_1\text{---}T_2)\text{---}h(T_{m-1}\text{---}T_m)$$

where X_i denotes x with subscript r_i

Y_i denotes y with subscript r_i

T_i denotes t with subscript r_i

and where $h(T_{i-1}\text{---}T_i)$ is a derived function of order $r_i - r_{i-1}$. The summation is over all non negative integers r_0, ---, r_m such that $o = r_0 < r_1 < \text{---} < r_m = n$.

It is the purpose of this paper to develop a corresponding formula for the nth derived function of a function of two functions. To do this we first define a type of derived function which we call

Partial Derived Functions.

Given a function $f(x;y)$ of two variables x and y, we define the function $f_{1,0}(x_0x_1;y_0)$ for distinct values of x_0 and x_1, by the equation

$$f_{1,0}(x_0x_1;y_0) = [f(x_1;y_0) - f(x_0;y_0)]/(x_1 - x_0)$$

Similarly, for distinct values of y_0 and y_1 we define $f_{1,1}(x_0x_1;y_0y_1)$ by the equation

$$f_{1,1}(x_0x_1;y_0y_1) = [f_{1,0}(x_0x_1;y_1) - f_{1,0}(x_0x_1;y_0)/y_1 - y_0)$$

Thus having defined the functions $f_{1,1}$ $f_{2,1}$, $f_{2,2}$, ---, $f_{n,m-1}$ for n+1 distinct x's and m distinct y's, we are able to define $f_{n,m}(x_0\text{---}x_n;y_0\text{---}y_m)$ by the equation

$$f_{n,m}(x_0\text{---}x_n;y_0\text{---}y_m) = [f_{n,m-1}(x_0\text{---}x_n;y_1\text{---}y_m) - f_{n,m-1}(x_0\text{---}x_n;y_0y_2\text{---}y_m)](y_1 - y_0)$$

The symmetry in the x's and in the y's of the function $f_{n,m}$ can be seen by the aid of the formula for expanding $f_{n,m}$ into partial fractions. We have

$$f_{n,m}(x_0 \cdots x_n;y_0 \cdots y_m) = \sum_{r=0}^{n} \sum_{s=0}^{m} \frac{f(x_r;y_s)}{p'(x_r)q'(y_s)}$$

where $p(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

$q(y) = (y-y_0)(y-y_1)\cdots(y-y_m)$.

This formula can be proved by ordinary induction and the proof is omitted.

The nth derived *f*unction o*f* a *f*unction o*f* two *f*unctions.

If $g(t)=f(x;y)$, $x=h(t)$, $y=j(t)$ we have by definition

$$g_1(t_0t_1)=[g(t_1)-g(t_0)]/(t_1-t_0)$$
$$=[f(x_1;y_1)-f(x_0;y_0)]/(t_1-t_0)$$
$$=[f(x_1;y_1)-f(x_0;y_1)]/(t_1-t_0)+[f(x_0;y_1)-f(x_0;y_0)]/(t_1-t_0)$$
$$=f_{1,0}(x_0x_1;y_1)(x_1-x_0)/(t_1-t_0)+f_{0,1}(x_0;y_0y_1)(y_1-y_0)/(t_1-t_0)$$
$$=f_{1,0}(x_0x_1;y_1)[h(t_1)-h\ (t_0)]/(t_1-t_0)$$
$$+f_{0,1}(x_0;y_0y_1)[j(t_1)-j)t_0)]/(t_1-t_0)$$
$$=f_{1,0}(x_0x_1;y_1)h_1(t_0t_1)+f_{0,1}(x_0;y_0y_1)j_1(t_0t_1)$$

We have thus expressed the Ampere-Cauchy derived function of the first order for a function of two functions in terms of our partial derived functions and regular derived functions.

To obtain the function of order 2, we proceed as before, being careful to add and subtract sufficient chosen terms to obtain a set of differences whose arguments differ only by a single subscript.

Thus $f_2(t_0t_1t_2)=[f_1(t_1t_2)-f(t_0t_2)]/(t_1-t_0)$

$$=[f_{1,0}(x_1x_2;y_2)h_1(t_1t_2)-f_{1,0}(x_0x_2;y_2)h_1(t_0t_2)$$
$$+f_{0,1}(x_1;y_1y_2)j_1(t_1t_2)-f_{0,1}(x_0;y_0y_2)j_1(t_0t_2)]/(t_1-t_0)$$
$$=[f_{1,0}(x_1x_2;y_2)-f_{1,0}(x_0x_2;y_2)]h_1(t_1t_2)/(t_1-t_0)$$
$$+[f_{1,0}(x_0x_2;y_2)[h_1(t_1t_2)-h_1(t_0t_2)]/(t_1-t_0)$$
$$+[f_{0,1}(x_1;y_1y_2)-f_{0,1}(x_0;y_1y_2)]j_1(t_1t_2)/(t_1-t_0)$$
$$+[f_{0,1}(x_0;y_1y_2)-f_{0,1}(x_0;y_0y_2)]j_1(t_1t_2)/(t_1-t_0)$$
$$+f_{0,1}(x_0;y_0y_2)[j_1(t_1t_2)-j_1(t_0t_2)]/(t_1-t_0)$$
$$=f_{2,0}(x_0x_1x_2;y_2)h_1(t_0t_1)h_1(t_1t_2)$$
$$+f_{1,0}(x_0x_2;y_2)h_2(t_0t_1t_2)$$
$$+f_{1,1}(x_0x_1;y_1y_2)h_1(t_0t_1)j_1(t_1t_2)$$
$$+f_{0,2}(x_0;y_0y_1y_2)j_1(t_0t_1)j_1(t_1t_2)$$
$$+f_{0,1}(x_0;y_0y_2)j_2(t_0t_1t_2)$$

Proceeding with expressions for $g_3\ (t_0t_1t_2t_3)$, etc., we discern the following

Formula: If $g(t)=f(x,y),\ x=h(t),\ y=j(t)$

$$g_n(t_0\text{---}t_n)=$$

$$\sum f_{m,q-m}(X_0\text{---}X_m;Y_m\text{---}Y_q)h(T_0\text{---}T_1)\text{---}$$

$$\text{---}h(T_{m-1}\text{---}T_m)j(T_m\text{---}T_{m+1})\text{---}$$

$$\text{---}j(T_{q-1}\text{---}T_q)$$

where X_i, Y_i, and T_i are so defined above and $h(T_{i-1}\text{---}T_i)$ and $j(T_{i-i}\text{---}T_i)$ are derived functions of order $r_i—r_{i-1}$ and where the summation is over all non negative integers r_0, r such that

$$0=r_0<\text{---}<r_q=n$$

The proof of this formula is by induction and we give only its outline. Referring to our results for $g_1(t_0t_1)$ and $g_2(t_0t_1t_2)$, we see that the formula holds for $n=1$, and $n=2$. Assuming its truth for $n=n$, we may write

$$g_{n+1}(t_0\text{---}t_{n+1})=[g_n(t_1\text{---}t_{n+1})—g_n(t_0t_2\text{---}t_{n+1})]/(t_1—t_0)$$

The nth ordered derived functions of the right hand member can be expressed by the formula, since its truth is assumed for nth ordered functions. We then proceed as in the case of $n=2$ to separate the results into differences which become higher ordered derived functions. In this way we obtain a result which proves the formula for $n=n+1$. This completes the inductive proof.

The fact that $f_{m,n}(x_0\text{---}x_m;y_0\text{---}y_n)$ approaches the partial derivative $f_{x^m y^n}(x,y)/(m!n!)$ leads to important geometrical applications of the partial derived functions.

ON PERIODIC FUNCTIONS. (PRELIMINARY INVESTIGATION)

By W. PAUL WEBBER
Louisiana State University

¶1. NOTATION. Define a variable vector by the vector equation,

(1) $\quad w=i_1x_1+i_2x_2+\text{----}+i_px_p$

where i_1, i_2 ----, i_p are unit vectors in the direction of a system of rectangular coordinates with axes OX_1, OX_2, ----, OX_p respectively in a space of p dimensions, and where x_1, x_2, ---, x_p

are the real coordinates of a point referred to these axes.

Define w the vector equation

(2) $\qquad w = 2i_1 m_1 w_1 + 2i_2 m_2 w_2 + \cdots + 2i_p m_p w_p$

where m_1, m_2, ----, m_p assume all positive and negative integer values, except that $m_1 = m_2 = \cdots = m_p = 0$ simultaneously, shall be omitted. The real quantities w_1, w_2, ----, w_p are arbitrary real constants.

It will be assumed that the ordinary rules of vector arithmetic carry over to p dimensions, so that if u_1, u_2 are two vectors defined by (1),

$$u_1 \cdot u_2 = |u_1|\,|u_2| \cos (u_1 u_2)$$
$$u_1 \times u_2 = |u_1|\,|u_2| \sin (u_1 u_2)$$

Also, will

$$u^2 = u : u \text{— and } u \times u = 0$$

In general even powers of u are scalars and odd powers vectors. Obviously,

$$|u| = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}$$
$$|u|^n = [\sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}]^n$$

The reciprocal of u will be defined as

$$\frac{1}{u} = \frac{1}{|u|} a$$

where a is a unit vector parallel to u. The ratio

$$\frac{u_1}{u} = \frac{|u_1|\,|a|}{|u|} \cos (u_1 a)$$

The formal existence of $u = 1 + u + \frac{u^2}{\lfloor 2} + \cdots + \frac{u^n}{\lfloor n} + \cdots$

and of log u will be assumed.

¶2. TWO GENERALIZED FUNCTIONS. Write the formal equation

(3) $\qquad S_p(u) = u \prod{}' \left[\left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} + \cdots + \frac{1}{p}\frac{u^p}{w^p}} \right.$

as a definition of a generalized Weierstrassian Sigma function, the accent on $\prod$ indicating that the factor corresponding to $w = 0$ is to be excluded.

Also, write the formal equation

$$(4)\qquad \operatorname{Log} S_p(u) = \log u + \Sigma'\left[\log\left(1-\frac{u}{w}\right)+\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}+\cdots+\frac{1}{p}\frac{u^p}{w^p}\right]$$

employing the derivative as a matter of form,

$$\frac{d}{du}\log S_p(u) = \frac{1}{u} + \Sigma'\left[\frac{1}{u-w}+\frac{1}{w}+\cdots+\frac{u^{p-1}}{w_p}\right]$$

and

$$\frac{d^p}{du^p}\log S_p(u) = {}^{p-1}\lfloor_p \left\{ \frac{1}{u^p} + \Sigma'\left[\frac{1}{(u-w)^p} - \frac{1}{u^p}\right] \right\}$$

Now define $F_p(u)$ as a generalized Weierstrassian pe-function by the equation,

$$(5)\qquad F_p(u) = \frac{(-1)^{p-1}}{\lfloor p-1}\left[\frac{d^p}{du^p}\log S_p(u)\right]$$

$$= \frac{(-1)^{p-1}}{\lfloor p-1}\left[\frac{1}{w_p} + \Sigma'\left(\frac{1}{(u-w)^p} - \frac{1}{w^p}\right)\right[$$

It can be shown that $F_p(u)$ is absolutely convergent for all finite values of u except $u=0$ and $u=w$, for all positive integral values of p.

$F_p(u)$ can be exhibited in form of the power series, convergent for (u) less than the distance from the origin and the nearest w point. For the particular case of p=3 in (3) and (5) there results

$$(6)\qquad F_3(u) = \frac{1}{u^3} + \Sigma'\left[\frac{1}{(u-w)^3} - \frac{1}{w^3}\right]$$

By letting p=3 in (4) and taking derivatives

$$(7)\, F^3(u) = \frac{1}{u^3} + \frac{3u}{w^4} + \frac{10u^3}{w^6} + \cdots + \frac{(n+2)(n+1)}{\lfloor 2 \quad w^{n+3}}u^n + \cdots$$

It is to be noticed that the coefficients in this series include

only even powers of w and $w=0$ excluded. This is owing to the symmetry of distribution of the w points about the origin, $n^n=-(-u)^n$ when u is odd. These coefficients are therefore scalar quantities, i. e. numbers. Since $F\ (u)$ will be a sum of terms of odd or of even degree acording as p is odd or even, it might be inferred $F_3(u)$ would contain only odd powers of u. Equation (7) shows this to be the case for p=3. Hence $F_3(u)$ is a vector quantity, each term being a vector.

The derivative of any function of u is not independent of the value of du. There will be a derivative in every direction at any point u. When only the form of the derivative is used it is indefinite in direction but has a definite magnitude at each point u. The indefinite derivative here indicated will be denoted by the usual accent. There will be a partial derivative in the direction of each component x_1 of u. The magnitude of du will be

$$|du|=\sqrt{dx_1^2+dx_2^2+\cdots+dx_p^2}$$

The direction cosines of du are

$$\cos a_1=\frac{dx_i}{|du|},\ i=1,\ 2,\ \text{---},\ p.$$

Resuming with $F_3(u)$

$$F_3'(u)=\frac{-3}{u^4}+\frac{3}{w^4}+\frac{30w^2}{w^6}+\cdots+\frac{(n+2)\ (u+1)n\,u^{n-1}}{\llcorner^2\ w^{n+1}}+\cdots$$

$F_3'(u)$ is a scalar quantity since its terms are all even powers of u with scalar coefficients. By reasoning similar to that employed in elliptic functions it is obvious that there exists an algebraic relation between $F_3(u)$ and $F_3(u)$ of degree three in $F_3'(u)$ and degree four in $F_3(u)$. This relation will be scalar since only even powers of u will occur. Similarly this property holds for $F_p(u)$ and $F_p'(u)$

$$\text{Taking } -\left[\frac{d^3}{du_3}\log S_3(u)\right]=(-1)^2\left[-\frac{d^2}{dw^2}\frac{S_3'(u)}{S_3(u)}\right]$$

and writing S_3 for $S_3(u)$ for brevity,

$$F_3(u)=\frac{(-1)}{\llcorner^2}\left[\frac{2\,S_3'^3-S_3'\,S_3''(2+S_3)-S_3^2\,S_3''}{S_3^3}\right]$$

It is now evident F (u) has the following properties:

1. Is one valued.

2. Is continuous for all points in space except the lattice points, w.

3. Has periods $2w$, $2w_2$, ----, $2w_2$.

4. Is an even function or an odd function acording as p is even or odd.

5. Is a scalar quantity when p is even and a vector when p is odd.

6. Satisfies an algebraic equation between $F_p(u($ and $F_p'(u)$ of degree p in $F_p'(u)$ and degree p+1 in $F_p(u)$.

7. Reduces to the same form as Wierstrass' p-function when p=2.

It is not to be supposed that $F_2(u)$ is an elliptic function. For $F_p(u)$ is defined for vector algebra while the elliptic functions are defined for ordinary complex algebra.

It is obvious that $F_p(u)$ is valid in any linear algebra that admits the necessary arithmetical processes for its formation. Under such a definition Weierstrass' theory of elliptic functions becomes a special case as do the trigonometric functions, etc.

The classic $S(u)$ is defined as

$$\frac{d}{du} \log S(u) = \frac{1}{u} + \Sigma'\left[\frac{1}{u-w} + \frac{1}{w} + \frac{u}{w^2}\right]$$

By analogy $z_3(u)$ may be defined as

$$z_3(u) = \frac{d^2}{du^2} \log S_3(u) = \frac{-1}{u^2} + \Sigma'\left[\frac{1}{(u-w)^2} + \frac{1}{w^2} + \frac{2w}{w^3}\right]$$

Substituting $u+2w_1$ for u

$$z_3(u+2w_1) = z_3(u) + 2\ z_3(w_1)$$

Similarly

$$z_3(u+2w_2) = z_3(u) + 2\ z_3(w_2)$$

and

$$z_3(u+2w_3) = z_3(u) + 2\ z_3(w_3)$$

In the case of p=4 the corresponding results are

$$z_4(u+2w_1) =$$

$$z_4(u) + 2z_4(w_1),$$

By substitution it is found

and $$S_4(w_1) = \Sigma' \frac{6w_1}{w^4}$$

and similarly for w_2, ----.

Read by title at September, 1927, meeting of American Mathematical Society at Chicago.

ON THE DIFFERENTIATION OF SCALAR FUNCTIONS OF VECTORS

By H. L. SMITH,
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In this paper the notion of derivative of a scalar function of a variable vector will be defined and some of the rules for computing with it will be established.

1. **The directional derivative.** Let $f(\mathrm{v})$ be a scalar function of a variable vector v and let u be a unit vector (i. e., a vector whose scalar square $\mathrm{u} \cdot \mathrm{u} = \mathrm{u}^2$ is 1). Let s be a scalar parameter and set

$$g(s) = f(\mathrm{v} + s\mathrm{u}).$$

Then the *derivative of f (v) in the direction* u is defined by the equation

$$D_{\mathrm{u}} f(\mathrm{v}) = g'(0).$$

Directly from this definition and the rules for differentiation of the calculus follow the folowing formulas:

(1) $D_{\mathrm{u}}\,[kf(\mathrm{v})] = k\,D_{\mathrm{u}}\,f(\mathrm{v})$,

(2) $D_{\mathrm{u}}\,[f_1(\mathrm{v}) + f_2(\mathrm{v})] = D_{\mathrm{u}}\,f_1(\mathrm{v}) + D_{\mathrm{u}}\,f_2(\mathrm{v})$,

(3) $D_{\mathrm{u}}\,[f_1(\mathrm{v}) f_2(\mathrm{v})] = f_1(\mathrm{v}) \cdot D_{\mathrm{u}}\,f_2(\mathrm{v}) + f_2(\mathrm{v}) \cdot D_{\mathrm{u}}\,f_1(\mathrm{v})$,

(4) $D_{\mathrm{u}}\,F(f(\mathrm{v})) = F'(f(\mathrm{v})) \cdot D_{\mathrm{u}}\,f(\mathrm{v})$,

and so on.

Now let i, j, k be three unit, mutually perpendicular vectors:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = 1, \quad \mathrm{i} \cdot \mathrm{j} = \mathrm{i} \cdot \mathrm{k} = \mathrm{j} \cdot \mathrm{k} = 0.$$

Then v, u can be expressed linearly in terms of i, j, k, say by the equations

(5) $\mathrm{v} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$

(6) $\mathrm{u} = a\mathrm{i} + b\mathrm{j} + c\mathrm{k}, \qquad (a^2 + b^2 + c^2 = 1).$

We then have

(7) $D_{\mathrm{u}}\,f(\mathrm{v}) = a\,D_x\,f(\mathrm{v}) + b\,D_y\,f(\mathrm{v}) + c\,D_z\,f(\mathrm{v})$.

By taking a, b, c to have successively the values 1, 0, 0, the values 0, 1, 0, and finally the values 0, 0, 1, we get

(8) $D_i f(v) = D_x f(v)$, $D_j f(v) = D_y f(v)$, $D_k f(v) = D_z f(v)$.

Hence (7) may be written

(9) $D_u f(v) = a\, D_i f(v) + b\, D_j f(v) + c\, D_k f(v)$.

Now let us set

(10) $r = [D_i f(v)]i + [D_j f(v)]j + [D_k f(v)]k$

Then (9) may be written

(11) $D_u f(v) = u \cdot r$.

We now have the following **theorem. For a given vector v there exists uniquely a unit vector u such that** $D_u f(v)$ **is a maximum. This vector is given by the formula**

(12) $u = r/\sqrt{r^2}$,

and for this value of u,

(13) $D_u f(v) = \sqrt{r^2}$,

where r is given by (10).

The truth of this theorem follows from the equation of definition of $u \cdot r$,

$u \cdot r = |u| \cdot |r| \cos \angle\, u \cdot r = |r| \cos \angle\, ur$, and from the fact that $\cos \angle ur$ has the value 1 if, and only if, (12) is satisfied.

2. **The vector derivative.** The (vector) derivative of $f(v)$. is now defined by the equation

(14) $Df(v) = [D_u f(v)]u$,

where u is the unique unit vector given by (12).

From (10), (12), (13), (14), it follows that

(15) $Df(v) = r = [D_i f(v)]i + [D\ f(v)]j + [D_k f(v)]k$.

From (15), (1), (2), (3), (4) we get at once

(16) $D[kf(v)] = k\, Df(v)$

(17) $D[f_1(v) + f_2(v)] = Df_1(v) + Df_2(v)$

(18) $D[f_1(v) \cdot f_2(v)] = f_1(v) \cdot Df_2(v) + f_2(v) \cdot Df_1(v)$.

(19) $DF(f(v)) = F'(f(v)) \cdot Df(v)$.

Now suppose v is a function of a scalar t. Then

(20) $D_t f(v) = Df(v) : D_t v$,

where D_t denotes differentiation with respect to t. This follows from (5), (8), (15) and the equation

$D_t f(v) = D_x f(v) \cdot D_t x + D_y f(v) \cdot D_t y + D_z f(v)\ D_t z$.

As an example we have

$Dv^2 = [D_x (x^2 + y^2 + z^2)]i + [D_y (x^2 + y^2 + z^2)]j + [D_z (x^2 + y^2 + z^2)]k$
$= 2xi + 2yj + 2zk = 2v$.

INFINITESIMAL AND FINITE INTEGRATION OF $(1/x)$.

By MRS. YETTA V. MAIZLISH
Centenary College

In the infinitesimal calculus we know that the formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

holds for all values of n except when $n = -1$. When n equals -1, we have

$$\int x^n dx = \frac{x^{n+1} - 1}{n+1} + c'$$

Since the right hand member of this equation is indeterminate when n equals -1, we differentiate numerator and denominator to evaluate the indeterminate form. The result obtained is

$$\int x^n dx = x^{n+1} \log x + c'$$

and when we set n equal to -1, we obtain

$$\int \frac{dx}{x} = \log x + c'$$

We find a similar situation in the Calculus of Finite Differences in the summation of the factorial

$$x^{(-n)},$$

which is defined by the equation

$$x^{(-n)} = \frac{1}{x\ (x+1)\ (x+2)\ \ldots\ (x+n-1)},$$

and which reduces to $(1/x)$ when n equals 1. The formula

$$\Sigma\ x^{(-n)} = -\ \frac{x^{(-n+1)}}{n-1} + c$$

holds for all values of n except $n=1$. Using the method of the Infinitesimal Calculus, we obtain the following:

$$\Sigma\ x^{(-n)} = -\ \frac{x^{(-n+1)} - 1}{n-1} + c'$$

$$= -\frac{\dfrac{1}{x\ (x+1)\ .\ .\ .\ .\ (x+n-1)} - 1}{n-1} + c'$$

$$= -\frac{\dfrac{\Gamma\ (x)}{\Gamma\ (x+n-1)} - 1}{n-1} + c'$$

The right hand member of this equation reduces to the indeterminate form $0/0+c'$ when $n=1$, and by differentiating numerator and denominator with respect to n, we obtain

$$\Sigma\, x^{(-n)} = \frac{\Gamma\ (x)\ \dfrac{d}{dn}\ \Gamma\ (x+n-1)}{[\Gamma\ (x+n-1)\ ]^2} + c'$$

By the definition of the Gamma Function,

$$\Gamma\ (x+n-1) = \int_0^\infty z^{x+n-2} e^{-z} dz,$$

and therefore,

$$\frac{d}{dn}\ \Gamma\ (x+n-1) = \int_0^\infty z^{x+n-2} e^{-z} \log z\ dz,$$

By substituting in the above equation, we obtain

$$\Sigma\ x^{(-n)} = \frac{\Gamma\ (x)\ \int_0^\infty z^{x+n-2} e^{-z} \log z\ dz}{[\Gamma\ (x+n-1)\ ]^2} + c',$$

and setting $n=1$, we have

$$\Sigma\ 1/x = \frac{\Gamma\ (x)\ \int_0^\infty z^{x-1} e^{-z} \log z\ dz}{[\Gamma\ (x)\ ]^2} + c',$$

or

$$\Sigma\ 1/x = \frac{\int_0^\infty z^{1-x} e^{-z} \log z\, dz}{\Gamma\ (x)} + c'.$$

Since

$$\int_0^\infty z^{x-1} e^{-z} \log z\, dz = \frac{d}{dx}\Gamma\ (x),$$

$$\Sigma\ 1/x = \frac{\frac{d}{dx}\Gamma\ (x)}{\Gamma\ (x)} + c'$$

$$= \frac{d}{dx} \log \Gamma\ (x) + c'$$

EDUCATIONAL GENERALISTS VERSUS SUBJECT-MATTER SPECIALISTS

By DORA M. FORNO
New Orleans Normal School

A masterly article in the May 1930 issue of The Mathematics Teacher by Dr. William C. Bagley entitled, "Professors of Education and Their Academic Colleagues" contains much food for thought for professors of education and subject-matter specialists in all fields of knowledge. Dr. Bagley has shown very forcefully how professors of education, administrators, supervisors and research workers, whom he has called "educational generalists", are monopolizing the places at council tables where the policies and programs of American education are being formulated. The absence of subject-matter specialists at the council table or the limited representation is deplored.

The question is asked, "How can we account for the ever widening influence of the professors of education in the public schools and the increasingly restricted influence of the subject-matter specialist?" The author reviews some interesting facts regarding the increased enrollment in high schools and colleges. He points out that other countries have made elementary education universal but that our country is rapidly working to-

ward universal secondary education and something closely akin to universal higher education.

This development involved a new attitude toward teaching problems and curricular problems. The higher institutions in the past had a selected group of students who followed well-established and highly standardized materials. The professors in these institutions were not concerned with how they should impart the subject-matter, but solely with the content of the subject. The student who had not the mentality nor the inclination to learn was early eliminated. The author states that "the subject-matter specialist has reflected and still tends to reflect the ideals of selective education rather than the ideals of mass-education.'"

If the subject-matter specialist is to hold his place of influence in the educational field, he must become a teacher-scholar with all the aims and ideals of the research worker and take his place at the council tables where the policies and programs of American education are being formulated. The changing school population in the upper grade, high schools, and colleges have developed problems of changing curricula to fit the needs of these students. The needs for arousing interest in these differentiated curricula and suiting these curricula to individual differences are problems that are being handled by "educational specialists".

To every teacher whether of an elementary grade, high school, normal school, or college, I would recommend most heartily this article by Dr. Bagley. To each it will bring an inspiring message of the need of co-ordination between the subject-matter that we teach and the method by which that subject-matter must be psychologized and vitalized for the enlightenment of the masses.

MATHEMATICS VIEWED FROM OTHER FIELDS.

"To the influence of this paper ('On the Equilibrium of Heterogeneous Substances' by Josiaph W. Gibbs) it is largely due that chemistry, then an almost unmathematical science, has so developed that the mathematical equipment now required by the student of chemistry differs but little from that which is requisite for the student of physics."

From an address by H. B. Williams, on "Mathematics and the Biologi-

cal Sciences," as published in May-June Bulletin of American Mathematical Society, 1925.

"It is less well understood that not alone the chemist and the physicist, but the biologist as well must be able to read mathematical papers if he is not to be cut off from the possibility of understanding important communications in his own field of science."—H. B. Williams.

"There are probably not a few professional mathematicians whose inherited capacity (in mathematics) is actually less than that of men who have quit the subject in despair. * * * * It seems that the lesson to be derived is that awakening of a real interest on the part of the student in his subject of study is quite as necessary for his success as is the possession of innate capacity."—H. B. Williams.

"As for the importance of the calculus, let us reiterate that science, engineering, and industry are demanding more and more calculus and that America the greatest manufacturing and engineering nation in the world, needs more trained mathematical minds; as a powerful mathematical tool, it shows the student the far-reaching influences of mathematics upon which the development of civilization has always been dependent; it develops a kind of thought in its dealings with small changes in related quantities that is useful in considering everyday problems."

From Susie B. Farmer's "The Place and Teaching of Calculus in Secondary Schools," as published in April **Mathematics Teacher.**

"Within the last few years there has been a great expansion of the use of mathematical methods in the field of economics. The expansion has taken two well-defined directions: one, that of statistical analysis of time series, the other that of laying a mathematical foundation for the science of economics."—By C. C. Morris, Ohio State University in Am. Mathematical Monthly Monthly, 1924.

"The greatest advance in men's mastery of the waves dates from the time that the hull of a ship was conceived as a geometric magnitude and was accordingly submitted to mathematical analysis."—By Commander Chantry, in address to Maryland-Virgina Section of M. A. of A., 1924.

"The main influence in the development of courses in mathematical statistics has come from the demand for better statistical methods in handling the data of the rapidly growing statistical sciences."—By H. L. Rietz, University of Iowa, Am. Math. Monthly, 1923.

"It is the purpose of this paper to indicate briefly the extent and types of mathematical analysis found in present day biology. The value

of such a summary is apparent to any adviser of students. This adviser realizes that he must not only recommend mathematics to the student but also show him the necessity of overcoming his aversion to mathematics in order to understand the experimental literature in biology and medicine." —By O. W. Richards, D'p't of Zoology, Univ. of Oregon, Am. Math. Monthly, 1925.

"Biology has, of necessity, taken over with the experimental methods of physics and chemistry the mathematical methods of description and analaysis which alone are capable of dealing with quantitatively measured variables and which, in consequence, early became an integral part of these two higly developed physical sciences."—By J. Arthur Harris, D'p't of Botany, University of Minnesota, Am. Math. Monthly, 1929.

"To see these things (in study of nature) requires more than a mere mathematician but the ablest mind which has never gone through a course in mathematics has small chance of ever seeing them."—By John Stuart Mill.

PROBLEM: FOR SOLUTION

By I. MAIZLISH

Problem 1. A wire, in the form of a cardioid, is to be covered with a ribbon whose width is w. If the ribbon is wound so that it overlaps by an amount e, find the minimum length of ribbon necessary.

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